

Continuum percolation of polydisperse hyperspheres in infinite dimensions

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We analyze the critical connectivity of systems of penetrable d -dimensional spheres having size distributions in terms of weighed random geometrical graphs, in which vertex coordinates correspond to random positions of the sphere centers and edges are formed between any two overlapping spheres. Edge weights naturally arise from the different radii of two overlapping spheres. For the case in which the spheres have bounded size distributions, we show that clusters of connected spheres are tree-like for $d \rightarrow \infty$ and they contain no closed loops. In this case we find that the mean cluster size diverges at the percolation threshold density $\eta_c \rightarrow 2^{-d}$, independently of the particular size distribution. We also show that the mean number of overlaps for a particle at criticality z_c is smaller than unity, while $z_c \rightarrow 1$ only for spheres with fixed radii. We explain these features by showing that in the large dimensionality limit the critical connectivity is dominated by the spheres with the largest size. Assuming that closed loops can be neglected also for unbounded radii distributions, we find that the asymptotic critical threshold for systems of spheres with radii following a lognormal distribution is no longer universal, and that it can be smaller than 2^{-d} for $d \rightarrow \infty$.

I. INTRODUCTION

Percolation phenomena are ubiquitous in many aspects of natural, technological, and social sciences, and they arise when system-spanning clusters or components of, in some sense, connected objects form [1, 2]. A quantity of much interest is the percolation threshold, which marks the transition between the phase in which a giant component exists and the one in which it does not. In general, the percolation threshold is a nonuniversal quantity, as it depends on the connectivity properties of the specific system under consideration [3]. For example, in continuum percolation systems, where objects occupy positions in a continuous space, the threshold depends on the shape of the objects [4–8], on their interactions [9–11], as well as on the connectedness criteria [12, 13].

In this article, we consider the infinite-dimensional limit of a paradigmatic example of continuum percolation: the Boolean-Poisson model [14, 15]. In this model, penetrable spheres with distributed radii have centers generated by a point Poisson process, and any two spheres are considered connected if they overlap. For a given distribution of the radii, the percolation threshold is given by the critical concentration η_c of spheres, or by the critical volume fraction $\phi_c = 1 - e^{-\eta_c}$, such that a giant component of connected spheres first forms. Precise numerical estimates of η_c have been obtained in two and three dimensions for, respectively, disks and spheres with fixed or distributed radii [16–23]. The general trend observed by these investigations is that η_c depends on the form of the distribution function of the radii, and that it has its minimum when the sphere radii are monodisperse (i.e., when the spheres have identical size). This last point has been formally confirmed in Ref. [24], although it may not hold true in the limit of infinite dimensions d [25, 26].

Here we show that for bounded distributions of the radii, that is for polydisperse spheres with a maximum finite value of the radius, the percolation threshold of the

Boolean-Poisson model tends asymptotically to a universal constant as $d \rightarrow \infty$, provided that the radii distribution is independent of d . This constant coincides with the value found in Refs. [27, 28] for spheres of identical radii, $\eta_c \rightarrow 2^{-d}$, and it is independent of the particular form of the size distribution function. We interpret the universality of η_c as being due to the statistical irrelevance of the spheres with smaller radii: the onset of percolation is established effectively only by the subset of spheres with maximum radius. Furthermore, we show that the mean number of connected spheres per particle at percolation, z_c , is less than unity for polydisperse distributions of the radii, while $z_c \rightarrow 1$ only in the limit of identical radii. This finding is analogous to what simulations have shown for the case of continuum percolation in three dimensional space of spherocylinders with length polydispersity [29].

These results rest on the observation that closed loops of connected spheres can be neglected in the limit of large dimensions, as we show explicitly for the case of bounded radii distributions. In the hypothesis that closed loops are irrelevant also for spheres of unbounded size, we show that η_c for $d \rightarrow \infty$ is not universal, as it depends on the parameters of the distribution, and that it can be smaller than the critical threshold of monodisperse spheres, in contrast to what is expected for finite dimensions [24].

II. THE MODEL

To construct the Boolean model, we consider N points placed independently and uniformly at random in a d -dimensional volume V . Each point is the center of a sphere with the radius drawn independently and randomly from a given probability distribution function $\rho(R)$. If we denote N_1 the number of spheres of radius R_1 , N_2 the number of spheres of radius R_2 , and so on,

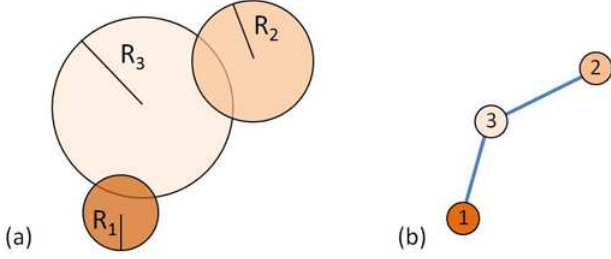


FIG. 1. (Color online) Connectedness criterion for spheres with different radii. (a) The spheres of radii R_1 and R_2 overlap the sphere of radius R_3 , forming links between R_3 and R_1 and between R_3 and R_2 . (b) Corresponding cluster formed by nodes (sphere centers) labeled by the sphere radii and weighted links (solid lines) connecting the nodes.

we can write the following without loss of generality:

$$\rho(R) = \sum_i x_i \delta(R - R_i), \quad (1)$$

where $x_i = N_i/N$ with $i = 1, 2, \dots$ is the fraction of spheres of radius R_i .

Given any two spheres of radii, say, R_i and R_j , we assign a link between their centers if the spheres overlap, that is, if the distance r between their center is smaller than $R_i + R_j$, as shown in Fig. 1. We express this criterion for the formation of a link in terms of the connectedness function:

$$f_{ij}(r) = \theta(R_i + R_j - r), \quad (2)$$

where $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$ is the unit step function.

The set of sphere centers (nodes) and links (edges) forms a type of weighted random geometric graph, in which the probability that an edge between two nodes is formed is weighted by the sphere radii. To see this, let us take a sphere of radius R_i centered at the origin. The probability that a second sphere of radius R_j forms a link with the first sphere is:

$$v_{\text{ex}}^{ij} = \frac{1}{V} \int d\mathbf{r} f_{ij}(r) = \Omega_d \frac{(R_i + R_j)^d}{V}, \quad (3)$$

where $d\mathbf{r}$ is an infinitesimal d -dimensional volume element at the position \mathbf{r} of the sphere of radius R_j , $\Omega_d = \pi^{d/2}/\Gamma(1 + d/2)$ is the volume of a sphere of unit radius, and Γ is the gamma function. We note that v_{ex}^{ij} defines also the excluded volume $V_{\text{ex}}^{ij} = \Omega_d (R_i + R_j)^d$ in units of V between two spheres of different radii.

III. IRRELEVANCE OF CLOSED LOOPS FOR $d \rightarrow \infty$

An important aspect of the topology of random geometric graphs is represented by closed loops (or cycles)

of connected nodes. The most studied loop quantity is the three-nodes cycle $c_d^{(3)}$, often denoted the cluster coefficient, which gives the conditional probability that two nodes are connected given that both nodes are connected to a third one. $c_d^{(3)}$ has been calculated for systems of spheres with identical radii and for any dimensionality [28, 30]. The observation that $c_d^{(3)}$ vanishes exponentially as $d \rightarrow \infty$ indicates that random geometric graphs in large dimensions have a locally tree-like structure.

Using results from the theory of hard-sphere fluids, it is actually possible to show that, in the limit of large dimensions, closed loops are negligible also for any number of nodes and for bounded radii distributions. Random and weighted random geometric graphs have thus tree-like structures when $d \rightarrow \infty$. To see this, let us first consider the case of monodisperse spheres with radius R_M . We define an n -chain graph as a cluster of $n \geq 3$ nodes with $n - 1$ edges such that every two consecutive edges, and only those, have a common node. We denote as end-nodes the two nodes of an n -chain that each have only one edge. The n -cycle coefficient $c_d^{(n)}$ is defined as the conditional probability that two nodes are connected given that they are the end-nodes of an n -chain. Since the spheres have identical radii, we omit the subscripts in Eq. (2), and we write the connectedness function as simply $f(r) = \theta(2R_M - r)$. From the definition of $c_d^{(n)}$, we can thus write:

$$c_d^{(n)} = \frac{\int d\mathbf{r}^{(n)} f(|\mathbf{r}_1 - \mathbf{r}_2|) f(|\mathbf{r}_2 - \mathbf{r}_3|) \cdots f(|\mathbf{r}_n - \mathbf{r}_1|)}{\int d\mathbf{r}^{(n)} f(|\mathbf{r}_1 - \mathbf{r}_2|) f(|\mathbf{r}_2 - \mathbf{r}_3|) \cdots f(|\mathbf{r}_{n-1} - \mathbf{r}_n|)}, \quad (4)$$

where $d\mathbf{r}^{(n)} = d\mathbf{r}_1 d\mathbf{r}_2 \cdots d\mathbf{r}_n$. Besides a prefactor, the above expression coincides with the cluster integral of a ring of n hard-spheres of radius R_M [31], as the Mayer function $f_M(r)$ for a fluid of hard-spheres is just $f_M(r) = -f(r)$ [3, 32]. To evaluate Eq. (4) for $d \rightarrow \infty$, we thus use known results from the theory of hard-sphere fluids in infinite dimensions. Noting that the denominator of Eq. (4) (i.e., the n -chain contribution) is simply $V V_{\text{ex}}^{n-1}$ [33], where $V_{\text{ex}} = \Omega_d 2^d R_M^d$ is the excluded volume for spheres of identical radius R_M , and introducing the Fourier transform $\hat{f}(\mathbf{q})$ of the connectedness function we rewrite Eq. (4) as:

$$c_d^{(n)} = \frac{1}{V_{\text{ex}}^{n-1}} \int \frac{d\mathbf{q}}{(2\pi)^d} \hat{f}(\mathbf{q})^n. \quad (5)$$

The integration in Eq. (5) for $d \rightarrow \infty$ has been worked out in Ref. [31] (see also Ref.[34]), so that the n -cycle coefficient reduces to:

$$c_d^{(n)} \rightarrow \sqrt{\frac{n-2}{\pi d(n-1)}} \left(\frac{n}{n-2} \right)^{n/2} \left[\frac{n^{n-2}}{(n-1)^{n-1}} \right]^{d/2}, \quad (6)$$

from which we see that closed loops of any number n of nodes are exponentially small as $d \rightarrow \infty$, because the quantity within square brackets is less than unity for $n \geq 3$.

Let us now consider the n -cycle coefficient for the case of polydisperse spheres. Using Eq. (2) for the connectedness function, we generalize Eq. (4) as follows:

$$\langle c_d \rangle^{(n)} = \frac{\langle C_{i_1, \dots, i_n}^{(n)} \rangle_{i_1, \dots, i_n}}{\langle \mathcal{V}_{i_1, \dots, i_n}^{(n)} \rangle_{i_1, \dots, i_n}}, \quad (7)$$

where

$$C_{i_1, \dots, i_n}^{(n)} = \int dr^{(n)} f_{i_1 i_2}(|\mathbf{r}_1 - \mathbf{r}_2|) f_{i_2 i_3}(|\mathbf{r}_2 - \mathbf{r}_3|) \cdots \cdots \times f_{i_n i_1}(|\mathbf{r}_n - \mathbf{r}_1|), \quad (8)$$

$$\mathcal{V}_{i_1, \dots, i_n}^{(n)} = \int dr^{(n)} f_{i_1 i_2}(|\mathbf{r}_1 - \mathbf{r}_2|) f_{i_2 i_3}(|\mathbf{r}_2 - \mathbf{r}_3|) \cdots \cdots \times f_{i_n i_1}(|\mathbf{r}_n - \mathbf{r}_1|), \quad (9)$$

and

$$\langle (\cdots) \rangle_{i_1, \dots, i_n} = \sum_{i_1, \dots, i_n} x_{i_1} x_{i_2} \cdots x_{i_n} (\cdots) \quad (10)$$

denotes a multiple average over the radii $R_{i_1}, R_{i_2}, \dots, R_{i_n}$. In the appendix, we show that for bounded distributions of radii, the n -cycle coefficient in the limit $d \rightarrow \infty$ is such that:

$$\langle c_d \rangle^{(n)} \leq c_d^{(n)} \chi_d^{(n)}, \quad (11)$$

where $c_d^{(n)}$ is the n -cycle coefficient for identical radii, Eq. (6), and $\chi_d^{(n)} \propto d^a$, where a is a nonnegative constant. Since the exponential decay of $c^{(n)}$ for $d \rightarrow \infty$ is stronger than the power-law increase of $\chi_d^{(n)}$, we see thus that also for the case of polydisperse spheres for bounded radii distributions, the n -cycle coefficient vanishes for any $n \geq 3$.

IV. SIZE OF FINITE COMPONENTS

The observation made in the previous section that closed loops are irrelevant in the large dimensional limit of the Boolean model allows us to consider the components of the associated weighted random geometric graph as effectively having a tree-like structure. This leads to a considerable simplification, as we can take the formalism of the theory of random graphs (see, e.g., Refs. [35–37]) and generalize it to the case in which nodes have weights.

A. Multidegree distributions

We start by considering the multidegree distribution of a node of type i , defined as the probability $p_i(1, k_1; 2, k_2; \dots)$ that a sphere of radius R_i is connected to k_1 spheres of radius R_1 , k_2 spheres of radius R_2 , and so on. Since the radii are randomly and independently distributed among the N nodes, $p_i(1, k_1; 2, k_2; \dots)$

is just a product of binomial distributions $p_{ij}(k_j)$ (with $j = 1, 2, \dots$), each giving the probability that k_j spheres of radius R_j overlap the sphere of radius R_i :

$$p_i(1, k_1; 2, k_2; \dots) = \prod_j p_{ij}(k_j), \quad (12)$$

with

$$p_{ij}(k_j) = \binom{N_j - \delta_{i,j}}{k_j} (v_{\text{ex}}^{ij})^{k_j} (1 - v_{\text{ex}}^{ij})^{N_j - \delta_{i,j} - k_j}, \quad (13)$$

where N_j (with $j = 1, 2, \dots$) is the number of spheres of radius R_j , v_{ex}^{ij} are the overlap probabilities given in Eq. (3), and $\delta_{i,j}$ is the Kronecker symbol.

We next consider for all i the limit $N_i \rightarrow \infty$ such that $N_i/V = x_i \rho$ remains finite, where $\rho = N/V$ is the total number density. In this limit, Eq. (13) reduces to a Poisson distribution:

$$p_{ij}(k_j) = \frac{z_{ij}^{k_j}}{k_j!} e^{-z_{ij}}, \quad (14)$$

where

$$z_{ij} = \sum_k k p_{ij}(k) = x_j \rho \Omega_d (R_i + R_j)^d \quad (15)$$

is the average number of spheres with radius R_j that overlap a given sphere of radius R_i .

In addition to the node degree distribution $p_i(1, k_1; 2, k_2; \dots)$, for the following analysis we will also need the excess node degree distribution $q_{ji}(1, k_1; 2, k_2; \dots)$, defined as the conditional probability that a sphere of radius R_j is connected to k_l spheres of radius R_l (with $l = 1, 2, \dots$), given that it is connected to a sphere of radius R_i . This task is simplified by the irrelevance of closed loops in the large dimensionality limit. In this case, indeed, if we select at random an edge connecting a node of type j with a node of type i , the j node attached to the edge is k_i times more likely to have degree k_i than degree 1 with nodes of type i . Its degree distribution will thus be proportional to $k_i p_j(1, k_1; 2, k_2; \dots)$. The excess degree distribution of a j node that has k_i edges with nodes of type i other than the edge with the node i to which is attached is thus [38]:

$$q_{ji}(1, k_1; 2, k_2, \dots) = \frac{(k_i + 1) p_j(1, k_1; \dots; i, k_i + 1; \dots)}{\sum_{\mathbf{k}} (k_i + 1) p_j(1, k_1; \dots; i, k_i + 1; \dots)}, \quad (16)$$

where $\sum_{\mathbf{k}} = \sum_{k_1, k_2, \dots}$. From Eqs. (12) and (14), $q_{ji}(1, k_1; 2, k_2, \dots)$ reduces simply to:

$$\begin{aligned} q_{ji}(1, k_1; 2, k_2, \dots) &= \frac{(k_i + 1) p_{ji}(k_i + 1)}{z_{ji}} \prod_{l \neq i} p_{jl}(k_l) \\ &= \prod_l p_{jl}(k_l), \end{aligned} \quad (17)$$

where we have used $(k_i + 1) p_{ji}(k_i + 1) = z_{ji} p_{ji}(k_i)$. Equation (17) states thus the well-known property that the excess degree distribution coincides with the node degree distribution when this is Poissonian [35].

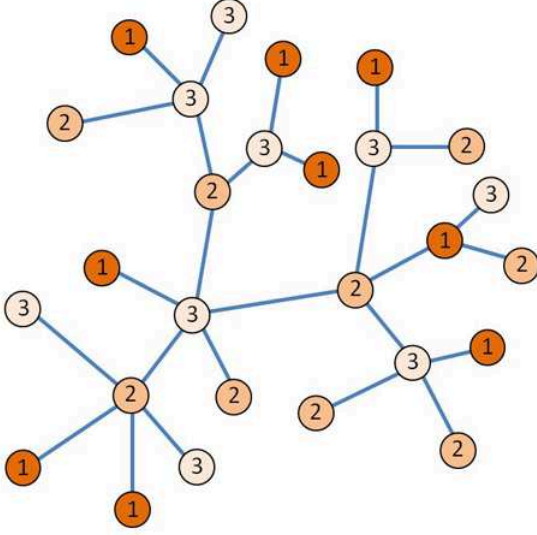


FIG. 2. (Color online) Schematic representation of a finite tree-like cluster formed by connected spheres of radii R_1 , R_2 , and R_3 . Each node label corresponds to the value of the radius of the sphere attached to the node.

B. Mean cluster size in the subcritical regime

We exploit now the statistical irrelevance of closed loops discussed in Sec. III to find the mean size S of finite clusters of connected spheres as $d \rightarrow \infty$. In doing so, we shall first keep the form of the degree distributions unspecified, and apply Eqs. (14) and (17) only at the end of the calculation.

Let us start by considering a randomly selected node that has probability x_i of being occupied by a sphere of radius R_i . Due to the general tree-like structure of the graph, the cluster to which the selected node belongs is formed by branches attached to the node according to the degree distribution $p_i(1, k_1; 2, k_2; \dots)$, as schematically shown in Fig. 2. The mean size S_i of the cluster to which the selected node belongs is thus:

$$S_i = x_i + x_i \sum_{\mathbf{k}} p_i(1, k_1; 2, k_2; \dots) \sum_j k_j T_{ij}, \quad (18)$$

where T_{ij} is the mean cluster size of one of the k_j branches attached to the selected node. Since the clusters have a tree-like structure, T_{ij} is given by the mass (unity) of one neighbor of the selected node, plus the mean cluster size of each of the remaining subbranches attached to the neighbor. To find T_{ij} , we thus need the excess degree distribution $q_{ji}(1, k_1; 2, k_2; \dots)$ of a sphere of radius R_j connected to the selected node of type i :

$$T_{ij} = 1 + \sum_{\mathbf{k}} q_{ji}(1, k_1; 2, k_2; \dots) \sum_l k_l T_{jl}. \quad (19)$$

Equations (18) and (19) are quite general, as they apply also to tree-like graphs with degree distributions that

are not reducible to a multiplication of Poissonian probabilities. Interestingly, similar equations are found in the calculation of finite size components of multigraphs (also denoted multiplex networks), formed by different networks, each having particular node properties, coupled together [38, 39]. The Boolean-Poisson model with random radii can thus be viewed also as a particular type of multigraph, in which each individual network is constituted by nodes occupied by spheres of a given radius.

Let us now use the results of Sec. IV A and rewrite Eqs. (18) and (19) by substituting $p_i(1, k_1; 2, k_2; \dots)$ and $q_{ji}(1, k_1; 2, k_2; \dots)$ with, respectively, Eqs. (12) and (17):

$$S_i = x_i + x_i \sum_j \sum_k k p_{ij}(k) T_j = x_i + x_i \sum_j z_{ij} T_j, \quad (20)$$

$$T_j = 1 + \sum_l \sum_k k p_{jl}(k) T_l = 1 + \sum_l z_{jl} T_l, \quad (21)$$

where we have used Eq. (15) and the fact that T_{ij} depends only on the neighbor (j) of the selected node, i.e., $T_{ij} = T_j$.

The mean cluster size is given by $S = \sum_i S_i$, which from Eq. (20) reduces to: $S = 1 + \sum_{ij} x_i z_{ij} T_j$. This relation is obtained also if we multiply both sides of Eq. (21) by x_j and sum over j . We can thus write:

$$S = \sum_j x_j T_j, \quad (22)$$

which states that S is just the average over the sphere radii of the mean cluster size of the branches.

C. Equivalence with the Ornstein-Zernike equation for the pair-connectedness

In continuum percolation theory, cluster statistics are often studied using the formalism of pair-connectedness correlation functions [3, 40, 41], which exploits well developed techniques of liquid state theory. This method has been recently used to studying percolation of monodisperse spheres in large dimensions [28].

As long as closed loops can be neglected, the network formalism discussed above and the pair-connectedness functions method give identical results, provided that the second-virial approximation is taken. To see how this equivalence holds true for the Boolean model in large dimensions, let us first consider the pair-connectedness function $P_{ij}(\mathbf{r} - \mathbf{r}')$, defined such that $x_i x_j \rho^2 P_{ij}(\mathbf{r} - \mathbf{r}') d\mathbf{r} d\mathbf{r}'$ is the probability of finding two spheres of radii R_i and R_j within the volume elements $d\mathbf{r}$ and $d\mathbf{r}'$ centered respectively in \mathbf{r} and \mathbf{r}' , given that they belong to the same cluster. The mean cluster size S is given in terms of $P_{ij}(\mathbf{r} - \mathbf{r}')$ by the following relation[42]:

$$S = 1 + \rho \sum_{i,j} x_i x_j P_{ij}, \quad (23)$$

where $P_{ij} = \int d\mathbf{r} P_{ij}(\mathbf{r})$. P_{ij} is the solution of the pair connectedness analog of the Ornstein-Zernike equation of the liquid state theory of fluids:

$$P_{ij} = C_{ij} + \rho \sum_l x_l C_{il} P_{lj}, \quad (24)$$

where $C_{ij} = \int d\mathbf{r} C_{ij}(\mathbf{r})$ is the volume integral of the direct pair connectedness function $C_{ij}(\mathbf{r})$, which describes short-range connectivity correlations. Let us introduce the quantity \tilde{T}_i defined as:

$$\tilde{T}_i = 1 + \rho \sum_j x_j P_{ij}. \quad (25)$$

The use of the above expression reduces Eq. (23) to:

$$\begin{aligned} S &= \sum_i x_i + \rho \sum_{i,j} x_i x_j P_{ij} = \sum_i x_i \left(1 + \rho \sum_j x_j P_{ij} \right) \\ &= \sum_i x_i \tilde{T}_i, \end{aligned} \quad (26)$$

while inserting Eq.(24) into Eq. (25) leads to:

$$\begin{aligned} \tilde{T}_i &= 1 + \rho \sum_j x_j \left(C_{ij} + \rho \sum_l x_l C_{il} P_{lj} \right) \\ &= 1 + \rho \sum_j x_j C_{ij} + \rho \sum_j x_j C_{ij} (\tilde{T}_j - 1) \\ &= 1 + \rho \sum_j x_j C_{ij} \tilde{T}_j. \end{aligned} \quad (27)$$

We see that Eqs. (26) and (27) are identical to respectively Eqs. (22) and (21) if we identify $\rho x_j C_{ij}$ with z_{ij} . From Eq. (15), we obtain thus:

$$C_{ij} = \frac{z_{ij}}{\rho x_j} = \Omega_d (R_i + R_j)^d, \quad (28)$$

which corresponds to take the volume integral of the second-virial approximation $C_{ij}(\mathbf{r}) = C_{ij}^{(2)}(\mathbf{r}) = f_{ij}(r)$ for the direct pair-connectedness function. This is not surprising, because in the density expansion of the direct pair-connectedness function, $C_{ij}(\mathbf{r}) = \sum_{n \geq 2} \rho^{n-2} C_{ij}^{(n)}(\mathbf{r})$, the terms with $n \geq 3$ contain at least one closed loop.

V. UNIVERSALITY OF THE PERCOLATION THRESHOLD

We proceed to find the percolation threshold for the Boolean-Poisson model of polydisperse spheres in the large dimensionality limit. We shall consider the case of bounded distributions of the radii, for which we have shown in Sec. III that closed loops of connected particles can be neglected for $d \rightarrow \infty$, and Eqs. (21) and (22) are

valid. To measure the sphere concentration we introduce the dimensionless density

$$\eta = \rho \Omega_d \langle R^d \rangle_R = \rho \Omega_d \sum_i x_i R_i^d. \quad (29)$$

The percolation threshold η_c is defined as the smallest value of η such that S diverges. This definition is equivalent to finding the smallest pole of Eq. (21), if it exists.

A. Discrete radii distributions

We first consider the case in which the spheres have a finite number M of radii:

$$\rho(R) = \sum_{i=1}^M x_i \delta(R - R_i), \quad (30)$$

so that using Eqs. (15), (21) and (22) we rewrite the equations for the mean cluster size as:

$$S = \sum_{i=1}^M x_i T_i, \quad (31)$$

$$T_i = 1 + \rho \Omega_d \sum_{j=1}^M x_j (R_i + R_j)^d T_j. \quad (32)$$

Without loss of generality, we assume that R_M is strictly the largest radius out of the M possible values of the radii, and we introduce $q_i = R_i/R_M$, which takes values smaller than the unity for all $i \neq M$. For large d , the dimensionless density η reduces to:

$$\begin{aligned} \eta &= \rho \Omega_d \sum_{i=1}^M x_i R_i^d = \rho \Omega_d R_M^d \left[x_M + \sum_{i=1}^{M-1} x_i q_i^d \right] \\ &\rightarrow \rho \Omega_d R_M^d x_M, \end{aligned} \quad (33)$$

because q_i^d goes exponentially to zero as $d \rightarrow \infty$ when $i \neq M$, and Eq. (32) becomes:

$$T_i = 1 + 2^d \eta \frac{1}{x_M} \sum_j x_j \left(\frac{q_i + q_j}{2} \right)^d T_j. \quad (34)$$

We note that $\left(\frac{q_i + q_j}{2} \right)^d$ is vanishingly small as $d \rightarrow \infty$ unless $i = j = M$, for which it takes the value 1. The smallest pole of Eq. (34) for large d is thus the solution of:

$$T_i = 1 + 2^d \eta T_M \delta_{i,M}, \quad (35)$$

where $\delta_{i,j}$ is the Kronecker delta. Equation (35) is solved by $T_M = 1/(1 - 2^d \eta)$ and $T_i = 1$ for $i \neq M$, so that the mean cluster size (31) becomes:

$$S = \sum_{i=1}^{M-1} x_i + x_M T_M = \frac{x_M}{1 - 2^d \eta}, \quad (36)$$

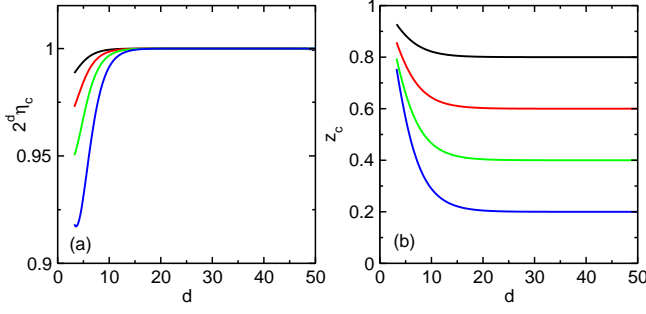


FIG. 3. (Color online) (a) Percolation threshold η_c in units of the asymptotic value 2^{-d} as a function of dimensionality for a discrete distribution of radii with $M = 2$ and $R_1/R_2 = 1/2$. In this case Eq. (32) is a system of two linear equations so that η_c can be calculated exactly for any d . $x_2 = 0.8, 0.6, 0.4$, and 0.2 from the uppermost to the lowermost curves. (b) Corresponding values of the critical coordination number z_c . As $d \rightarrow \infty$, z_c tends asymptotically to x_2 .

which diverges when

$$\eta \rightarrow \eta_c = \frac{1}{2^d}. \quad (37)$$

The above expression for η_c holds true for any sequence of occupation fractions x_i , independent of dimensionality, provided that $x_M \neq 0$. In particular, Eq. (37) confirms and extends to $M > 2$ the finding of a previous report that spheres with two different radii have a universal critical threshold in infinite dimensions [26]. Note that $\eta_c = 1/2^d$ is also the limit for infinite dimensions of the percolation threshold of monodisperse spheres with radius R_M , whose mean cluster size is given by Eq. (36) with $x_M = 1$.

The origin of the universality of η_c can be traced back to the divergence of T_M , which indicates that the onset of a giant component of connected polydisperse spheres is established only by the subset of spheres with the maximum radius when $d \rightarrow \infty$. In other words, at $d \rightarrow \infty$ the contribution to percolation of the smaller spheres vanishes, and the resulting η_c is the critical threshold for a system of monodisperse spheres of radius R_M . Following the observation that systems of polydisperse spheres with M different radii can be interpreted as a multinet-work of coupled M subnetworks (see Sec. IV B), we see that Eq. (35) is equivalent to decoupling the subnetworks associated with each radius, and that long-range connectivity arises only from the network formed by spheres with radius R_M .

One interesting consequence of the irrelevance of smaller radii at percolation is that the critical average connectivity per particle,

$$z_c = \sum_{i,j} x_i x_j \rho_c \Omega_d (R_i + R_j)^d, \quad (38)$$

reduces for $d \rightarrow \infty$ to:

$$z_c = 2^d \rho_c \Omega_d R_M^d \sum_{i,j} x_i x_j \left(\frac{q_i + q_j}{2} \right)^d \rightarrow 2^d \eta_c x_M = x_M, \quad (39)$$

where ρ_c is the critical number density. For $x_M < 1$, the critical average connectivity is thus less than the unity for $d \rightarrow \infty$, which must be contrasted to $z_c \geq 1$ for systems constituting only of monodisperse spheres in any dimension [28].

For the binary case ($M = 2$), Eq. (32) reduces to a system of two linear equations that can be solved exactly for any d . The resulting η_c and z_c are shown in Figs. 3(a) and 3(b), respectively, for $R_2 = 2R_1$ and different values of the fraction x_2 of spheres of radius R_2 . The asymptotic limits $\eta_c = 2^{-d}$ and $z_c = x_2$ are recovered for sufficiently large values of d .

B. Continuous radii distributions

Let us now consider the case in which the radii distribution $\rho(R)$ is a continuous bounded function independent of d . We again denote by $R_M < \infty$ the maximum allowed radius, so that $\rho(R) = 0$ for $R > R_M$, and we rewrite the equations for the mean cluster size in terms of continuous variables of the radii:

$$S = \langle T(R) \rangle_R, \quad (40)$$

$$T(R) = 1 + \rho \Omega_d \langle (R + R')^d T(R') \rangle_{R'}, \quad (41)$$

where $\langle (\dots) \rangle_R = \int_0^{R_M} dR \rho(R) (\dots)$. We expand the binomial power $(R + R')^d$ and use $\eta = \rho \Omega_d \langle R^d \rangle_R$ to write:

$$T(R) = 1 + \eta \sum_{k=0}^d \binom{d}{k} \frac{R^{d-k}}{\langle R^d \rangle_R} \langle R^k T(R) \rangle_R. \quad (42)$$

If we multiply both sides of Eq. (42) by $R^n / \langle R^n \rangle_R$, with $n = 1, 2, \dots, d$, and average over R we arrive at:

$$t(n) = 1 + \eta \sum_{k=0}^d \binom{d}{k} \frac{\langle R^{n+d-k} \rangle_R \langle R^k \rangle_R}{\langle R^d \rangle_R \langle R^n \rangle_R} t(k), \quad (43)$$

where

$$t(n) = \frac{\langle R^n T(R) \rangle_R}{\langle R^n \rangle_R}. \quad (44)$$

From Eqs. (40) and (44) we see that the mean cluster size can be obtained from $S = t(0)$.

To solve Eq. (43), we note that for large d the binomial coefficient is strongly peaked at $k = d/2$ and takes the asymptotic form

$$\binom{d}{k} \simeq 2^d \sqrt{\frac{2}{\pi d}} e^{-\frac{2}{d}(k-d/2)^2} = \frac{2^{d+1}}{d} g\left(\frac{2k}{d} - 1, \frac{1}{\sqrt{d}}\right), \quad (45)$$

where $g(x, \sigma) = \exp(-x^2/2\sigma^2)/\sqrt{2\pi\sigma^2}$ is the Gaussian function. Provided that the radii distribution is bounded, the binomial coefficient dominates the k -dependence of the kernel. To see this, let us consider the m -th moment $\langle R^m \rangle_R = R_M^m \int_0^1 dy \rho(y) y^m$, where $y = R/R_M$. For large m the main contribution to the integral comes from y close to 1. Thus we make the quite general assumption that for $y \rightarrow 1$, the radii distribution behaves as $\rho(y) \propto (1-y)^{\alpha-1}$, with $\alpha > 0$. Setting $t = m(1-y)$ for large m , we find

$$\begin{aligned} \langle R^m \rangle_R &\propto \frac{R_M^m}{m^\alpha} \int_0^m dt t^{\alpha-1} (1-t/m)^m \\ &\simeq \frac{R_M^m}{m^\alpha} \int_0^\infty dt t^{\alpha-1} e^{-t} = \frac{R_M^m}{m^\alpha} \Gamma(\alpha), \end{aligned} \quad (46)$$

so that for large k the term $\langle R^{n+d-k} \rangle_R \langle R^k \rangle_R$ in Eq. (43) is proportional to $R_M^{n+d}/[(n+d-k)k]^\alpha$, which has a much weaker k -dependence than Eq. (45). Next, we introduce $s = 2n/d$ and $s' = 2k/d$, which we treat as continuous variables for $d \rightarrow \infty$, and we replace the sum over k by an integral over s' : $\sum_{k=0}^d \rightarrow \frac{d}{2} \int_0^2 ds'$. If we denote $\tilde{t}(s) = t(ds/2)$ and $\tilde{t}(s') = t(ds'/2)$, Eq. (43) becomes:

$$\begin{aligned} \tilde{t}(s) = 1 + 2^d \eta \int_0^2 ds' \left[g\left(s' - 1, \frac{1}{\sqrt{d}}\right) \right. \\ \left. \times \frac{\langle R^{d[1+(s-s')/2]} \rangle_R \langle R^{ds'/2} \rangle_R}{\langle R^d \rangle_R \langle R^{ds/2} \rangle_R} \tilde{t}(s') \right]. \end{aligned} \quad (47)$$

Since $g(s' - 1, 1/\sqrt{d}) \rightarrow \delta(s' - 1)$ for $d \rightarrow \infty$, the above expression reduces to:

$$\tilde{t}(s) = 1 + \eta 2^d \frac{\langle R^{d(1+s)/2} \rangle_R \langle R^{d/2} \rangle_R}{\langle R^d \rangle_R \langle R^{ds/2} \rangle_R} \tilde{t}(1), \quad (48)$$

from which we obtain the mean cluster size:

$$S = \tilde{t}(0) = 1 + \eta 2^d \frac{\langle R^{d/2} \rangle_R^2}{\langle R^d \rangle_R} \tilde{t}(1). \quad (49)$$

Setting $s = 1$ in Eq. (48), we find $\tilde{t}(1) = (1 - 2^d \eta)^{-1}$, so that we arrive finally at:

$$S = \frac{1}{1 - 2^d \eta} \frac{\langle R^{d/2} \rangle_R^2}{\langle R^d \rangle_R}, \quad (50)$$

which, as found for the case of discrete distributions, diverges at

$$\eta \rightarrow \eta_c = \frac{1}{2^d}, \quad (51)$$

independently of the particular form of the bounded distribution $\rho(R)$.

Using Eq. (45) and considering the weak dependence of the moments of R , we readily obtain the large dimen-

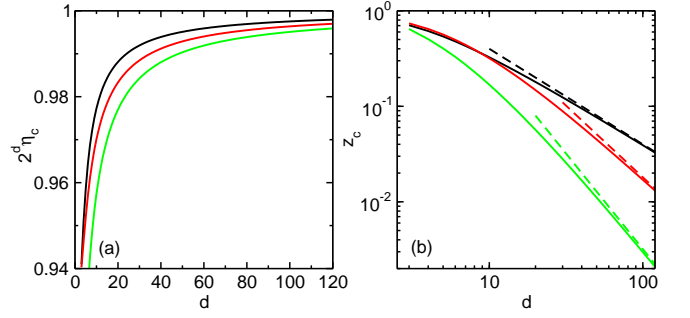


FIG. 4. (Color online) (a) Dimensional dependence of the percolation threshold η_c in units of the asymptotic value 2^{-d} obtained from a numerical solution of Eq. (43) for rectangular (upper curve), semicircular (middle curve), and triangular (lower curve) distributions of the sphere radii. (b) Corresponding critical average connectivity per particle z_c (solid curves). Dashed lines are the asymptotic results for $d \gg 1$: $z_c = 4/d$ (rectangular distribution), $z_c = 32/(\sqrt{\pi} d^{3/2})$ (semicircular distribution), and $z_c = 32/d^2$ (triangular distribution).

sional limit of the critical average connectivity per particle:

$$\begin{aligned} z_c = \rho_c \Omega_d \langle (R + R')^d \rangle_{R, R'} = \eta_c \sum_{k=0}^d \binom{d}{k} \frac{\langle R^k \rangle_R \langle R^{d-k} \rangle_R}{\langle R^d \rangle_R} \\ \rightarrow \frac{\langle R^{d/2} \rangle_R^2}{\langle R^d \rangle_R}, \end{aligned} \quad (52)$$

from which we see that $z_c \leq 1$ for any bounded distribution of the radii. Note that from Eq. (52) we recover $z_c = x_M$ when $\rho(R)$ is given by Eq. (30).

We complete this section by showing how the percolation threshold obtained from Eqs. (40) and (41) evolves towards the asymptotic value $\eta_c = 2^{-d}$ as d increases. Toward that end, we consider radii distributions of rectangular, semicircular, and triangular shapes, given respectively by $\rho(R) = 1/R_M$, $\rho(R) = 4\sqrt{(R_M/2)^2 + (R - R_M/2)^2}/\pi$, and $\rho(R) = 2(R_M - R)/R_M^2$, for $R \leq R_M$ and zero otherwise. We calculate η_c from the smallest pole of $S = t(0)$ obtained from a numerical solution of Eq. (43). The resulting thresholds are very close to 2^{-d} for all d considered, and they approach the asymptotic limit from below, as shown in Fig. 4(a). For the same cases of Fig. 4(a), we have calculated also the d -dependence of z_c , shown in Fig. 4(b) by solid lines, which we compare with the asymptotic limits (dashed lines) $z_c = 4/d$, $z_c = 32/(\sqrt{\pi} d^{3/2})$, and $z_c = 32/d^2$ obtained from Eq. (52) for rectangular, semicircular, and triangular distributions of the radii, respectively.

VI. THE CASE OF UNBOUNDED DISTRIBUTION OF THE RADII

Having established that η_c is universal as $d \rightarrow \infty$ for bounded (and independent of d) distributions of the radii, it is natural to ask if universality holds true also when $\rho(R)$ is unbounded. Although we have shown the irrelevance of closed loops limited to the case of bounded distributions, we shall nevertheless assume that n -cycle coefficients are negligible also for unbounded $\rho(R)$, and that graph components have a tree-like structure. Let us consider the specific case of a lognormal distribution function:

$$\rho(R) = \frac{1}{\sqrt{2\pi}\sigma R} \exp\left[-\frac{\ln^2(R/R_0)}{2\sigma^2}\right], \quad (53)$$

where $R \in [0, \infty)$, R_0 is the median radius, and σ is the standard deviation of $\ln(R)$. Equation (53) is an interesting case-study, as the resulting η_c and z_c for asymptotically large d can be found analytically. Using the k -th moment $\langle R^k \rangle_R = R_0^k \exp(\sigma^2 k^2/2)$, Eq. (43) becomes:

$$t(n) = 1 + \eta \sum_{k=0}^d \binom{d}{k} e^{\frac{1}{2}\sigma^2[(n+d-k)^2 + k^2 - n^2 - d^2]} t(k), \quad (54)$$

from which we express the mean cluster size as:

$$S = t(0) = 1 + \eta \sum_{k=0}^d \binom{d}{k} e^{\sigma^2 k(k-d)} t(k). \quad (55)$$

For sufficiently large d , the only nonvanishing terms of the summation are those with $k = 0$ and $k = d$, so that:

$$S = 1 + \eta[S + t(d)], \quad (56)$$

where from Eq. (54) $t(d)$ is given by:

$$t(d) = 1 + \eta \sum_{k=0}^d \binom{d}{k} e^{\sigma^2(d-k)^2} t(k). \quad (57)$$

For $d \rightarrow \infty$, $t(d)$ tends asymptotically to $t(d) = 1 + \eta e^{\sigma^2 d^2} t(0)$, as the term with $k = 0$ dominates the sum over k in Eq. (57). We thus find that the mean cluster size, Eq. (56), reduces to:

$$S = \frac{1 + \eta}{1 - \eta^2 e^{\sigma^2 d^2}}, \quad (58)$$

which diverges at the asymptotical critical value,

$$\eta \rightarrow \eta_c = e^{-\frac{1}{2}\sigma^2 d^2}. \quad (59)$$

The corresponding critical coordination number is

$$\begin{aligned} z_c &= \eta_c \sum_{k=0}^d \binom{d}{k} \frac{\langle R^k \rangle_R \langle R^{d-k} \rangle_R}{\langle R^d \rangle_R} = \eta_c \sum_{k=0}^d \binom{d}{k} e^{\sigma^2 k(k-d)} \\ &\rightarrow 2\eta_c, \end{aligned} \quad (60)$$

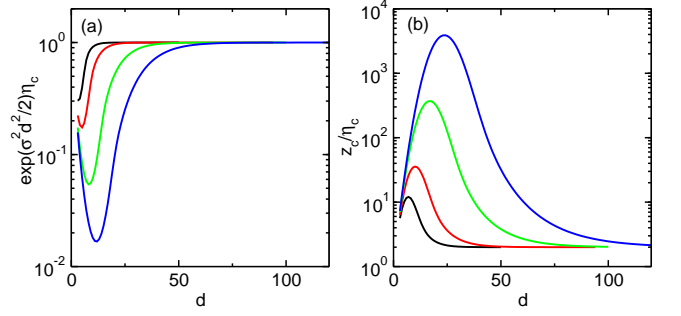


FIG. 5. (Color online) (a) Dimensional dependence of the percolation threshold η_c in units of the asymptotic value $\exp(-\sigma^2 d^2/2)$ for a lognormal distribution of the radii obtained from a numerical solution of Eq. (54); $\sigma = 0.25, 0.3, 0.4$, and 0.5 from the lowermost to the uppermost curves. (b) Critical average connectivity per particle z_c for $\sigma = 0.25, 0.3, 0.4$, and 0.5 from the uppermost to the lowermost curves. All curves tend to $z_c/\eta_c \rightarrow 2$ as $d \rightarrow \infty$.

where we have again used the fact that for large d only the terms $k = 0$ and $k = d$ contribute to the summation.

As evidenced in Eq. (59), the percolation threshold for infinite dimensions is no longer universal, as it depends on the parameter σ of the log-normal distribution. Interestingly, from Eq. (59) we also see that η_c can be smaller than the critical threshold of monodisperse spheres ($\eta_c = 2^{-d}$), contrary to what is expected in finite dimensions [24]. We note that a critical threshold smaller than the monodisperse sphere limit in large dimensions has been found also for the case of radii distributions with d -dependent weights [25, 26].

To verify the accuracy of Eq. (59), we compare it with the threshold obtained by solving numerically Eq. (54). As d increases, the asymptotic limit $\eta_c = e^{-\frac{1}{2}\sigma^2 d^2}$ is reached more rapidly when σ is larger, as shown in Fig. 4(a). From inspection of Eq. (55) we see that this behavior is due to the competition between $e^{\sigma^2 k(k-d)}$ and the maximum value $\sim 2^d$ of the binomial coefficient at $k = d/2$: the latter is suppressed by the exponential function when $d > 4 \ln(2)/\sigma^2$. From numerical calculation of z_c , shown in Fig. 5(b) for the same σ values of Fig. 5(a), we see that also the asymptotic formula for z_c , Eq. (60), is verified.

VII. LOWER BOUND ON THE PERCOLATION THRESHOLD

Having established that Eqs. (21) and (22) give asymptotic limits of the critical threshold η_c as $d \rightarrow \infty$, we show now that the same equations provide also a lower bound on η_c for any dimensionality. Toward that end, we take the pair-connectedness function $P_{ij}(\mathbf{r})$ considered in Sec. IV C, and we extend to the polydisperse sphere case

the inequality formulated in Ref. [43]:

$$P_{ij}(\mathbf{r}) \leq f_{ij}(r) + \rho \sum_l x_l \int d\mathbf{r}' f_{il}(|\mathbf{r} - \mathbf{r}'|) P_{lj}(\mathbf{r}'), \quad (61)$$

where $f_{ij}(r)$ is the connectedness function given in Eq. (2). The above expression applies to any dimensionality, and following Ref. [28], where Eq. (61) has been used for the monodisperse sphere case, it enable us to find a lower bound on the percolation threshold. To see this, we take the volume integral of Eq. (61),

$$P_{ij} \leq V_{\text{ex}}^{ij} + \rho \sum_l x_l V_{\text{ex}}^{il} P_{lj}, \quad (62)$$

where $V_{\text{ex}}^{ij} = \int d\mathbf{r} f_{ij}(r) = \Omega_d (R_i + R_j)^d$, and we use Eqs. (25) to find:

$$\tilde{T}_i \leq 1 + \rho \sum_j x_j V_{\text{ex}}^{ij} \tilde{T}_j = 1 + \sum_j z_{ij} \tilde{T}_j, \quad (63)$$

which together with Eq. (26) gives an upper bound for the mean cluster size:

$$S = \sum_i x_i \tilde{T}_i \leq \sum_i x_i T_i, \quad (64)$$

where T_i is the solution of Eq. (21). From the inequality of Eq. (64), we see that the value of η such that $\sum_i x_i T_i$ diverges identifies a lower bound on the percolation threshold for any d . The solid lines plotted in Figs. 3(a)-5(a) represent thus lower bounds on η_c for the different radii distribution functions considered in this work. As d increases, these lower bounds tend asymptotically to the infinite dimensional limit 2^{-d} for bounded radii distributions and to $\exp(-\sigma^2 d^2/2)$ for lognormal radii distributions. Finally, we note that Eq. (64) implies also that the values of z_c shown in Figs. 3(b)-5(b) are lower bounds on the critical connectivity for any dimensionality.

VIII. SUMMARY AND DISCUSSION

We have considered random dispersions of penetrable d -dimensional spheres with distributed radii in terms of weighted random geometric graphs, where nodes represent sphere centers and edges connect nodes of overlapping spheres with probability weighted by the sphere radii. For bounded distribution of the radii, we have shown that closed loops of connected spheres can be neglected in the limit $d \rightarrow \infty$ and that graph components have thus tree-like structure. Analysis of the mean cluster size reveals that the asymptotic percolation threshold is universal and coincides with the threshold $\eta_c = 2^{-d}$ found for the case of monodisperse spheres in high dimensions. This result confirms and extends a previous finding on the percolation of $d \rightarrow \infty$ spheres with two different radii [25, 26]. Furthermore, we show that the

asymptotic critical connectivity per particle z_c , though dependent on the shape of the radii distribution function, is less than unity and approaches $z_c \rightarrow 1$ for spheres of identical radii.

We have also studied critical connectivity for spheres with radii distributed according to a d -independent log-normal function, which is a treatable example of unbounded distribution. Assuming that clusters have a tree-like structure, we find that the percolation threshold η_c depends on the shape of the log-normal distribution and, interestingly, that η_c for $d \rightarrow \infty$ can be smaller than the threshold for monodisperse spheres, in contrast to what is expected at finite dimensions [24].

Before concluding, let us speculate on the percolation threshold in homogeneous fluids of polydisperse spheres with impenetrable cores (cherry-pit model [3]). In finite dimensions, correlations between the cores preclude writing the multi-degree distribution as a product of Poisson distributions, as done in Sec. IV A, because the N -particle distribution function $g_N(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$ depends on the relative positions of the core centers [32]. However, in the limit of infinite dimensions and for small densities, $g_N(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$ asymptotically factorizes into a product of θ -functions that are unity for pair distances beyond the hard-core diameter [44]. The multi-degree distribution for $d \rightarrow \infty$ can thus still be written as a product of Poisson distributions, with the average number of contacts unaltered by the presence of the hard-cores if the penetrable shells are non-vanishing. With the same reasoning, closed loops are expected to be negligible and graphs are still dominated by tree-like components. For non-zero penetrable shells, therefore we expect the same asymptotic results for η_c as those obtained for the case of penetrable hyperspheres.

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Appendix A: Irrelevance of $\langle c_d \rangle^{(n)}$ for $d \rightarrow \infty$

In this appendix, we show that when the radii distribution is independent of d and bounded [that is, when $\rho(R) = 0$ for any $R > R_M$, with $R_M < \infty$], the n -cycle coefficient for polydisperse spheres, defined in Eqs. (7)-(9), vanishes for $d \rightarrow \infty$.

Since R_M is the maximum radius of the distribution, the connectedness functions in the integrand of Eq. (8) are such that $f_{ij}(r) \leq f(r) = \theta(2R_M - r)$ for any i and j . We can thus write:

$$C_{i_1, \dots, i_n}^{(n)} \leq \int dr^{(n)} f(|\mathbf{r}_1 - \mathbf{r}_2|) f(|\mathbf{r}_2 - \mathbf{r}_3|) \cdots f(|\mathbf{r}_n - \mathbf{r}_1|), \quad (A1)$$

which, when substituted in Eq. (7), gives:

$$\langle c_d \rangle^{(n)} \leq c_d^{(n)} \frac{V V_{\text{ex}}^{n-1}}{\langle \mathcal{V}_{i_1, \dots, i_n}^{(n)} \rangle_{i_1, \dots, i_n}}, \quad (\text{A2})$$

where $V_{\text{ex}} = 2^d \Omega_d R_M^d$, and $c_d^{(n)}$ is the n -cycle coefficient for identical radii given in Eq. (6). Next, we perform the integrations over $\mathbf{r}_1, \dots, \mathbf{r}_n$ in Eq. (9) to find:

$$\begin{aligned} \mathcal{V}_{i_1, \dots, i_n}^{(n)} &= V \prod_{j=1}^{n-1} V_{\text{ex}}^{i_j, i_{j+1}} = V \Omega_d^{n-1} \prod_{j=1}^{n-1} (R_{i_j} + R_{i_{j+1}})^d \\ &= V \Omega_d^{n-1} \prod_{j=1}^{n-1} \left[\sum_{k_j=0}^d \binom{d}{k_j} R_{i_j}^{k_j} R_{i_{j+1}}^{d-k_j} \right], \end{aligned} \quad (\text{A3})$$

where in the last equality we have expanded the binomial powers. In performing the average over R_{i_1}, \dots, R_{i_n} , we must group the contributions with equal radius variables and average them independently of the other radii. De-

noting a general m -th moment as $\langle R^m \rangle_R$, we obtain:

$$\begin{aligned} \langle \mathcal{V}_{i_1, \dots, i_n}^{(n)} \rangle_{i_1, \dots, i_n} &= V \Omega_d^{n-1} \sum_{k_1=0}^d \binom{d}{k_1} \cdots \sum_{k_{n-1}=0}^d \binom{d}{k_{n-1}} \\ &\quad \times \langle R^{k_1} \rangle_R \langle R^{d-k_1+k_2} \rangle_R \langle R^{d-k_2+k_3} \rangle_R \cdots \\ &\quad \cdots \langle R^{d-k_{n-2}+k_{n-1}} \rangle_R \langle R^{d-k_{n-1}} \rangle_R. \end{aligned} \quad (\text{A4})$$

Following Sec. VB, we approximate for large d the binomial coefficients by Gaussian functions centered at $d/2$, and we replace the sums by integrals, so that for $d \rightarrow \infty$, $\sum_{k_j=0}^d \binom{d}{k_j} \rightarrow 2^d \int_0^2 ds_j \delta(s_j - 1)$, where $s_j = 2k_j/d$ and $j = 1, \dots, n-1$. Equation (A4) reduces in this way to:

$$\langle \mathcal{V}_{i_1, \dots, i_n}^{(n)} \rangle_{i_1, \dots, i_n} \rightarrow V \Omega_d^{n-1} 2^{(n-1)d} \langle R^{d/2} \rangle_R^2 \langle R^d \rangle_R^{n-2}, \quad (\text{A5})$$

so that Eq. (A2) becomes:

$$\langle c_d \rangle^{(n)} \leq c_d^{(n)} \chi_d^{(n)}, \quad (\text{A6})$$

where

$$\chi_d^{(n)} = \frac{R_M^{(n-1)d}}{\langle R^{d/2} \rangle_R^2 \langle R^d \rangle_R^{n-2}}. \quad (\text{A7})$$

For continuous radii distributions, we assume that $\rho(R) \propto (R_M - R)^{\alpha-1}$ with $\alpha > 0$ for $R \rightarrow R_M$, as done in Sec. VB. Using Eq. (46) we thus find $\chi_d^{(n)} \propto d^{n\alpha}$. For discrete distributions as in Eq. (30), it is straightforward to show from Eq. (A7) that $\chi_d^{(n)} \rightarrow 1/(x_M)^n$ for large d . We have thus arrived at the result that $\chi_d^{(n)}$ increases with d at most as a power-law, leading to $\langle c_d \rangle^{(n)} \rightarrow 0$ as $d \rightarrow \infty$, due to the exponential vanishing of $c_d^{(n)}$.

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